

# On Cotriangular Hopf Algebras

Pavel Etingof

Department of Mathematics, Rm 2-165

Massachusetts Institute of Technology

77 Massachusetts Avenue

Cambridge, MA 02139

Shlomo Gelaki

Mathematical Sciences Research Institute

1000 Centennial Drive

Berkeley, CA 94720

February 1, 2008

## 1 Introduction

In [EG1, Theorem 2.1] we proved that any triangular semisimple Hopf algebra over an algebraically closed field  $k$  of characteristic 0 is obtained from the group algebra  $k[G]$  of a finite group  $G$ , by twisting its comultiplication by a twist in the sense of Drinfeld [Dr]. Since semisimple Hopf algebras are finite-dimensional, dualizing yields that any cotriangular semisimple Hopf algebra over  $k$  is obtained from  $k[G]^*$ , the function algebra on  $G$ , by twisting its multiplication by a Hopf 2-cocycle in the sense of Doi [Do] (see Section 2 below).

In this paper we generalize Theorem 2.1 from [EG1] to not necessarily finite-dimensional cotriangular Hopf algebras  $A$  over  $k$ . Namely, our main result (see Section 3 below) is:

**Theorem** A cotriangular Hopf algebra  $A$  over  $k$  is obtained from the function algebra  $\mathcal{O}(G)$  of a pro-algebraic group  $G$ , by twisting its multiplication by a Hopf 2-cocycle, and possibly changing its R-form by a central grouplike element of  $A^*$  of order  $\leq 2$ , if and only if  $\mathrm{tr}(S^2|_C) = \dim(C)$  for any finite-dimensional subcoalgebra  $C$  of  $A$  (where  $S$  is the antipode of  $A$ ).

The main challenge in the proof of this theorem (see Section 4 below) is to establish the “if” direction. The key step in the proof of the “if” part is to show that our trace condition on  $A$  guarantees that the categorical dimensions of objects in the category of its

finite-dimensional right comodules are non-negative integers (maybe after modifying the R-form). This enables us to apply the same theorem of Deligne on Tannakian categories [De] that we applied in the proof of Theorem 2.1 from [EG1].

In Section 5, we give examples of twisted function algebras. In particular, we show that in the infinite-dimensional case, the squared antipode for such an algebra may not equal the identity (see Example 5.2 below).

In Section 6, we show that in all of our examples, the operator  $S^2$  is unipotent on  $A$ , and conjecture it to be the case for any twisted function algebra. We prove this conjecture, using the quantization theory of [EK1-2], in a large number of special cases.

In Section 7, we formulate a few open questions.

Throughout the paper,  $k$  will denote an algebraically closed field of characteristic 0.

**Acknowledgements** The first author is grateful to Ben Gurion University for its warm hospitality, and to Miriam Cohen and the Dozor Fund for making his visit possible; his work was also supported by the NSF grant DMS-9700477.

The second author is grateful to Susan Montgomery for numerous useful conversations.

The authors would like to acknowledge that this paper was inspired by the work [BFM].

## 2 Hopf 2-cocycles

Let  $A$  be a coassociative coalgebra over  $k$ . For  $a \in A$ , we write  $\Delta(a) = \sum a_1 \otimes a_2$ ,  $(I \otimes \Delta)\Delta(a) = \sum a_1 \otimes a_2 \otimes a_3$  etc, where  $I$  denotes the identity map of  $A$ .

Recall that  $A^*$  is an associative algebra with product defined by  $(f * g)(a) = \sum f(a_1)g(a_2)$ . This product is called the *convolution product*.

Now let  $(A, m, 1, \Delta, \varepsilon, S)$  be a Hopf algebra over  $k$ .

Recall [Do] that a linear form  $J : A \otimes A \rightarrow k$  is called a *Hopf 2-cocycle* for  $A$  if it has an inverse  $J^{-1}$  under the convolution product  $*$  in  $\text{Hom}_k(A \otimes A, k)$ , and satisfies:

$$\sum J(a_1 b_1, c) J(a_2, b_2) = \sum J(a, b_1 c_1) J(b_2, c_2) \text{ and } J(a, 1) = \varepsilon(a) = J(1, a) \quad (1)$$

for all  $a, b, c \in A$ .

Given a Hopf 2-cocycle  $J$  for  $A$ , one can construct a new Hopf algebra  $(A^J, m^J, 1, \Delta, \varepsilon, S^J)$  as follows. As a coalgebra,  $A^J = A$ . The new multiplication is given by

$$m^J(a \otimes b) = \sum J^{-1}(a_1, b_1) a_2 b_2 J(a_3, b_3) \quad (2)$$

for all  $a, b \in A$ . The new antipode is given by

$$S^J(a) = \sum J^{-1}(a_1, S(a_2)) S(a_3) J(S(a_4), a_5) \quad (3)$$

for all  $a \in A$ .

Suppose  $A$  is also co(quasi)triangular with universal R-form  $R : A \otimes A \rightarrow k$  (see e.g. [K, VIII.5.1]). Then it is straightforward to verify that  $A^J$  is co(quasi)triangular with universal R-form  $R^J : A^J \otimes A^J \rightarrow k$ , where  $R^J := (J \circ \tau)^{-1} * R * J$  (here  $\tau : A \otimes A \rightarrow A \otimes A$  is the usual flip map).

Recall [Dr] that a *twist* for a Hopf algebra  $B$  is an invertible element  $J \in B \otimes B$  which satisfies

$$(\Delta \otimes I)(J)(J \otimes 1) = (I \otimes \Delta)(J)(1 \otimes J) \text{ and } (\varepsilon \otimes I)(J) = (I \otimes \varepsilon)(J) = 1. \quad (4)$$

It is straightforward to verify that if  $A$  is finite-dimensional, then  $J \in A^* \otimes A^*$  is a Hopf 2-cocycle for  $A$  if and only if it is a twist for  $A^*$ .

### 3 The Main Theorem

Let  $(A, R)$  be a cotriangular Hopf algebra over  $k$  (not necessarily finite-dimensional). Define the Drinfeld element of  $(A, R)$  to be the linear form  $u : A \rightarrow k$  determined by

$$u(a) = \sum R(a_2, S(a_1)). \quad (5)$$

Recall that  $u \in A^*$  is a grouplike element, and that

$$S^2(a) = (u * I * u^{-1})(a) = \sum u(a_1) a_2 u^{-1}(a_3) \quad (6)$$

for all  $a \in A$ .

Suppose  $c \in A^*$  is a central grouplike element of order  $\leq 2$ , and set

$$R_c := \frac{1}{2}(\varepsilon \otimes \varepsilon + \varepsilon \otimes c + c \otimes \varepsilon - c \otimes c). \quad (7)$$

Then it is straightforward to verify that  $(A, R * R_c)$  is cotriangular with Drinfeld element  $u * c$ .

Note that by (6),  $S^2$  preserves any subcoalgebra of  $A$ .

**Definition 3.1** *We say that  $A$  is pseudoinvolutive if  $\text{tr}(S^2|_C) = \dim(C)$  for any finite-dimensional subcoalgebra  $C$  of  $A$ .*

**Remark 3.2** If  $A$  is finite-dimensional, then pseudoinvolutivity is equivalent to involutivity ( $S^2 = I$ ). Indeed, by [R],  $S^2$  has a finite order, so its eigenvalues on  $A$  are roots of 1. But a sum of  $\dim(A)$  roots of 1 can be equal to  $\dim(A)$  only if all of them are 1.  $\square$

We can now state our main theorem.

**Theorem 3.3** *Let  $(A, R)$  be a cotriangular Hopf algebra over  $k$ . Then the following two conditions are equivalent:*

(i)  *$A$  is pseudoinvolutive.*

(ii)  *$A$  is a twisted function algebra  $\mathcal{O}(G)^J$  on a pro-algebraic group  $G$ , and furthermore, there exists a central grouplike element  $c \in \mathcal{O}(G)^*$  of order  $\leq 2$ , such that  $(A, R)$  is isomorphic to  $(\mathcal{O}(G)^J, (J \circ \tau)^{-1} * R_c * J)$  as cotriangular Hopf algebras.*

**Remark 3.4** Theorem 3.3 is a generalization of Theorem 2.1 from [EG1]. Indeed, let  $A$  be a finite-dimensional triangular Hopf algebra over  $k$ . Equivalently,  $A^*$  is a finite-dimensional cotriangular Hopf algebra over  $k$ . Now, by Remark 3.2,  $A^*$  is pseudoinvolutive if and only if  $S^2 = I$ . By [LR], this is equivalent to the semisimplicity of  $A$  and  $A^*$ . Hence by Theorem 3.3,  $A^*$  is a finite-dimensional semisimple cotriangular Hopf algebra over  $k$  if and only if it is a twisted function algebra  $\mathcal{O}(G)^J$  on a pro-algebraic group  $G$  (possibly, with a changed cotriangular structure). But of course,  $G$  must be a finite group.  $\square$

**Remark 3.5** The data  $(G, c, J)$  corresponding to a pseudoinvolutive cotriangular Hopf algebra over  $k$ , is unique up to isomorphism of such triples and gauge transformations of  $J$  (see [EG2]). The proof is similar to the proof of Lemma 3.5 in [EG2].  $\square$

**Remark 3.6** If  $(A, R)$  is a cocommutative cotriangular Hopf algebra over  $k$  (hence also commutative), then Theorem 3.3 is applicable since in this situation  $S^2 = I$ . Thus, [BFM, Theorem 3.19(i)], which claims that in this case  $(A, R)$  is a twisted group algebra (maybe with  $R \rightarrow R * R_c$ ), is a special case of our result.  $\square$

**Remark 3.7** In [CWZ, Theorem 2.1] the authors prove that if  $(A, R)$  is cotriangular and its Drinfeld element  $u$  acts as the identity on a finite-dimensional right  $A$ -comodule  $V$ , then the characters of the usual action of the symmetric group  $S_n$  on  $V^{\otimes n}$  and the one arising from the braiding  $R$  are equal. We note that this result is a consequence of Theorem 3.3. Namely, one should apply the theorem to the cotriangular Hopf algebra  $H_V \subset A$  which is generated by the elements of the form  $(f \otimes I)(\rho_V(v))$ ,  $v \in V, f \in V^*$  (here  $\rho_V$  denotes the structure map of  $V$ ), and get that the symmetric category of finite-dimensional right comodules of the cotriangular Hopf algebra  $H_V$  is equivalent to that of  $\mathcal{O}(G)$  for some pro-algebraic group  $G$ . Since the characters of  $S_n$  on  $V^{\otimes n}$  are invariant under equivalences of symmetric categories, the result follows. We also see that Theorem 2.1 of [CWZ] can be strengthened by replacing the assumption  $u|_V = 1$  by the weaker assumption that  $u|_V$  is unipotent.  $\square$

## 4 The Proof of Theorem 3.3

### 4.1 Changing the Cotriangular Structure

In this subsection we prove the following proposition, which is one of the key ingredients in the proof of Theorem 3.3.

**Proposition 4.1.1** *For any pseudoinvolutive cotriangular Hopf algebra  $(A, R)$  over  $k$ , there exists a central grouplike element  $c \in A^*$  of order  $\leq 2$  such that for the cotriangular Hopf algebra  $(A, R * R_c)$ , with Drinfeld element  $u$ , one has  $\text{tr}(u|_V) = \dim(V)$  for any finite-dimensional right  $A$ -comodule  $V$ .*

The rest of the subsection is devoted to the proof of this proposition.

Let  $(A, R)$  be a cotriangular Hopf algebra over  $k$ . Let  $\mathcal{C} := \text{Comod}_{f.d.}(A)$  be the category of finite-dimensional right  $A$ -comodules, and  $\text{Irr}(\mathcal{C})$  be the subcategory of all irreducible objects of  $\mathcal{C}$ . It is straightforward to check that  $\mathcal{C}$  is an abelian rigid symmetric tensor category in the sense of [DM]. The unit object of  $\mathcal{C}$  is  $\mathbf{1} := k$ , and clearly  $\text{End}(\mathbf{1}) = k$ . Also, recall that for any object  $V \in \mathcal{C}$ , one can define its *categorical dimension* [DM], denoted by  $\dim_c(V)$ , to be the image of 1 under the morphism  $k \rightarrow V \otimes V^* \rightarrow V^* \otimes V \rightarrow k$  (where the morphism  $V \otimes V^* \rightarrow V^* \otimes V$  is the braiding map). It is straightforward to verify that

$$\dim_c(V) = \text{tr}|_V(u), \quad (8)$$

where  $u$  is regarded as the linear map  $V \rightarrow V$  determined by  $v \mapsto (I \otimes u)\rho_V(v)$  (where  $\rho_V$  denotes the structure map of  $V$ ). Observe that  $\text{tr}|_V(u) = \text{tr}|_V(u^{-1})$  for any  $V \in \mathcal{C}$ . Indeed, we have that  $u^{-1} = u \circ S$  and  $R = R \circ (S \otimes S)$ , hence  $u(a) = u^{-1}(a)$  for any cocommutative element  $a \in A$ . But,  $\text{tr}|_V \in A$  is a cocommutative element.

For any object  $V \in \mathcal{C}$ , set

$$A_V := \{(f \otimes I)\rho_V(v) | v \in V, f \in V^*\}. \quad (9)$$

It is clear that  $A_V$  is a finite-dimensional subcoalgebra of  $A$ .

From now on we assume that  $A$  is pseudoinvolutive.

**Definition 4.1.2** *We say that an object  $V \in \mathcal{C}$  is positive if  $\dim_c(V) = \dim(V)$ , and negative if  $\dim_c(V) = -\dim(V)$ .*

**Lemma 4.1.3** *An object of  $\text{Irr}(\mathcal{C})$  is either positive or negative, and any object of  $\mathcal{C}$  is positive (resp. negative) if and only if so are all the composition factors of its Jordan-Hölder series.*

**Proof:** Let  $X \in \text{Irr}(\mathcal{C})$ . Then  $A_X = X \otimes X^*$ . Since  $\text{tr}|_X(u) = \text{tr}|_X(u^{-1})$ , it follows from (6) that  $\dim(X)^2 = \text{tr}(S^2|_{A_X}) = (\text{tr}|_X(u))(\text{tr}|_X(u^{-1})) = \dim_c(X)^2$ . Thus,  $\dim_c(X) = \pm \dim(X)$  as desired. Moreover, for any  $V \in \mathcal{C}$ , if  $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$  is its Jordan-Hölder series, then  $\dim_c(V) = \sum_{i=1}^n \dim_c(V_i/V_{i-1})$ , and the result follows. ■

Consider now the abelian category of finite-dimensional right *bicomodules* over  $A$ , i.e. right comodules over  $A \otimes A^{\text{cop}}$ . It is clear that irreducible objects of this category are of the form  $X \otimes Y^*$ , where  $X, Y \in \text{Irr}(\mathcal{C})$ .

For any  $V \in \mathcal{C}$ , it is clear that  $A_V$  is a right  $A$ -bicomodule (recall that  $A_V^*$  is the image of  $A^* \rightarrow \text{End}(V)$ , so it is a left  $A^* \otimes A^{\text{op}}$ -module). For any  $X, Y \in \text{Irr}(\mathcal{C})$ , let  $N_V(X, Y)$  be the multiplicity of occurrence of  $X \otimes Y^*$  as a composition factor in the Jordan-Hölder series of  $A_V$  regarded as a right bicomodule over  $A$ .

**Lemma 4.1.4** *For any  $V \in \mathcal{C}$  and  $X, Y \in \text{Irr}(\mathcal{C})$  with opposite signs,  $N_V(X, Y) = 0$ .*

**Proof:** Indeed,

$$\begin{aligned} & \text{tr}(S^2|_{A_V}) \\ &= \sum_{X, Y \in \text{Irr}(\mathcal{C})} N_V(X, Y) \text{tr}|_{X \otimes Y^*} (u \otimes (u^{-1})^*) \\ &= \sum_{X, Y \in \text{Irr}(\mathcal{C})} N_V(X, Y) \dim_c(X) \dim_c(Y). \end{aligned}$$

But by pseudoinvolutivity, this should be equal to

$$\dim(A_V) = \sum_{X, Y \in \text{Irr}(\mathcal{C})} N_V(X, Y) \dim(X) \dim(Y).$$

This implies that if  $\dim_c(X)$  and  $\dim_c(Y)$  have opposite signs then  $N_V(X, Y) = 0$ , otherwise  $\sum_{X, Y \in \text{Irr}(\mathcal{C})} N_V(X, Y) \dim_c(X) \dim_c(Y) < \sum_{X, Y \in \text{Irr}(\mathcal{C})} N_V(X, Y) \dim(X) \dim(Y)$ . ■

**Lemma 4.1.5** *For any  $V, W \in \mathcal{C}$  with opposite signs, one has  $\text{Ext}^1(V, W) = 0$ .*

**Proof:** We have to show that any exact sequence  $0 \rightarrow V \rightarrow U \rightarrow W \rightarrow 0$  in  $\mathcal{C}$  splits. Indeed, we have natural coalgebra embeddings  $A_V \hookrightarrow A_U$ ,  $A_W \hookrightarrow A_U$  (generated by the coactions on the subcomodule and the quotient comodule), and the sum of them is a coalgebra embedding  $A_V \oplus A_W \hookrightarrow A_U$  (it is obvious that the sum is direct as the coalgebras  $A_V, A_W$  do not intersect, because of the opposite signs of  $V, W$ ). Consider the quotient  $A_U/(A_V \oplus A_W)$  as an  $A$ -bicomodule. Its all composition factors are of the form  $X_+ \otimes X_-^*$ , where  $X_+ \in \text{Irr}(\mathcal{C})$  is positive and  $X_- \in \text{Irr}(\mathcal{C})$  is negative, which is impossible by Lemma 4.1.4. Therefore, this bicomodule must be zero. Thus,  $A_V \oplus A_W \cong A_U$ . Let  $W' := \rho_U^{-1}(U \otimes A_W)$ . Then  $U = V \oplus W'$ , and we are done. ■

**Lemma 4.1.6** *Any object  $V \in \mathcal{C}$  can be uniquely represented as a direct sum  $V_+ \oplus V_-$ , where  $V_+ \in \mathcal{C}$  is positive and  $V_- \in \mathcal{C}$  is negative. Furthermore, for any two objects  $V, W \in \mathcal{C}$ ,  $(V \otimes W)_+ = (V_+ \otimes W_+) \oplus (V_- \otimes W_-)$ , and  $(V \otimes W)_- = (V_+ \otimes W_-) \oplus (V_- \otimes W_+)$ .*

**Proof:** We first prove the existence part of the lemma by induction in  $\dim(V)$ . Let  $V \in \mathcal{C}$ , and  $X \in \text{Irr}(\mathcal{C})$  be an irreducible subcomodule of  $V$ . Let us assume  $X$  is positive. By the induction assumption we have  $V/X = W_+ \oplus W_-$ . Let  $V_+$  be the preimage of  $W_+$  under the projection  $V \rightarrow W_+$ . Then  $V_+$  is positive, and  $V/V_+ = W_-$ . Therefore, by Lemma 4.1.5,  $V$  is isomorphic to  $V_+ \oplus W_-$ , and the result follows. If  $X$  is negative, the proof is similar.

We now prove the uniqueness part of the lemma. Let  $V = V_+ \oplus V_-$ . Then it is easy to see that  $V_+$  is the sum of all positive subcomodules of  $V$ , and  $V_-$  is the sum of all negative subcomodules, which implies uniqueness.

The last statement of the lemma is obvious. ■

Let  $D$  be any coalgebra over  $k$ , let  $\text{Comod}(D)$  be its category of right comodules and  $F : \text{Comod}(D) \rightarrow \text{Vec}$  be the forgetful functor. Recall that  $\text{End}(F) \cong D^*$  as algebras. Indeed, an element  $\eta \in \text{End}(F)$  is by definition, a collection of linear maps  $\eta_V : V \rightarrow V$ ,  $V \in \text{Comod}(D)$ , which commute with comodule morphisms. In particular,  $\eta_D : D \rightarrow D$  is a linear map between right  $D$ -comodules, and it commutes with right actions by elements of  $D^*$ . Thus,  $\eta_D$  comes from a left action by an element of  $D^*$  (namely, by  $\eta_D^*(\varepsilon)$ ). Note that if moreover  $\eta_V : V \rightarrow V$  is a comodule map for all  $V \in \text{Comod}(D)$ , then we have an endomorphism of the identity functor  $Id$  of  $\text{Comod}(D)$ , and the resulting element of  $D^*$  is central. Thus,  $\text{End}(Id) \cong \text{Center}(D^*)$  as algebras.

For any  $V \in \mathcal{C}$ , define the comodule automorphism  $c_V$  of  $V$  by  $c_V := I$  on  $V_+$  and  $c_V := -I$  on  $V_-$ , where  $I$  is the identity map of  $V$ .

**Lemma 4.1.7** *The collection  $\{c_V | V \in \mathcal{C}\}$  determines a central grouplike element  $c \in A^*$  of order  $\leq 2$ .*

**Proof:** The collection  $\{c_V | V \in \mathcal{C}\}$  is an element of the algebra  $\text{End}(Id)$ , hence, by the preceding remarks, determines a central element  $c \in A^*$ . Now, it is clear from Lemma 4.1.6 that  $c$  is a grouplike element of order  $\leq 2$ . ■

Now let us finally prove Proposition 4.1.1. Let  $c \in A^*$  be the central grouplike element of order  $\leq 2$  whose existence is guaranteed by Lemma 4.1.7. Let  $R_c$  be as in (7). Then, after changing  $R$  to  $R * R_c$ , the new Drinfeld element is  $u' := u * c$ , and we get that  $\text{tr}|_V(u') = \dim(V)$  for any object  $V \in \mathcal{C}$ . The proposition is proved. ■

## 4.2 The Proof of Theorem 3.3

(i)  $\Rightarrow$  (ii).

Proposition 4.1.1 implies that without loss of generality, we can assume that  $\text{tr}(u|_V) = \dim(V)$  for all finite-dimensional right  $A$ -comodules.

Now comes the main step of the proof, which is the usage of the following theorem of Deligne.

**Theorem 4.2.1 (De, Theorem 7.1)** *Let  $\mathcal{C}$  be an abelian rigid symmetric tensor category over  $k$  such that  $\text{End}(\mathbf{1}) = k$ , in which categorical dimensions of objects are non-negative integers. Then there exist a pro-algebraic group  $G$  and a  $k$ -linear equivalence of abelian rigid symmetric tensor categories  $F : \mathcal{C} \rightarrow \text{Rep}_{f.d.}(G)$  (where  $\text{Rep}_{f.d.}(G)$  is the category of finite-dimensional algebraic  $k$ -representations of  $G$ ).*

This theorem implies that in our situation, we have an equivalence

$$F : \text{Comod}_{f.d.}(A) \rightarrow \text{Comod}_{f.d.}(\mathcal{O}(G))$$

of rigid symmetric tensor categories. It is obvious that  $F$  preserves dimensions.

Now we will need the following proposition, whose proof occupies the next subsection.

**Proposition 4.2.2** *Let  $A$  and  $B$  be two coassociative coalgebras with counit over  $k$ , and  $F : \text{Comod}_{f.d.}(A) \rightarrow \text{Comod}_{f.d.}(B)$  be an equivalence between the abelian categories of finite-dimensional right comodules over  $A$  and  $B$ , which preserves dimensions. Then there exists an isomorphism of coalgebras  $\phi : A \rightarrow B$  such that  $F$  is isomorphic to the direct image functor  $\phi_*$ .*

**Remark 4.2.3** In the case when  $A, B$  are cosemisimple, this proposition is trivial. Therefore, it was not spelled out explicitly in our previous papers, where we dealt exclusively with the semisimple case.  $\square$

This proposition implies that there exists an isomorphism of coalgebras  $\phi : A \rightarrow \mathcal{O}(G)$  that induces  $F$ . Therefore, we can naturally identify the vector spaces  $V$  and  $F(V)$  for all  $V \in \mathcal{C}$ , in a functorial way.

Now, recall that  $F$  has a tensor structure. Namely, we have a collection of right  $\mathcal{O}(G)$ -comodule isomorphisms  $J_{VW} : F(V) \otimes F(W) \rightarrow F(V \otimes W)$  indexed by all pairs  $V, W \in \mathcal{C}$ . But for any  $U \in \mathcal{C}$ , we have already identified the vector spaces  $U$  and  $F(U)$ . Thus, we can regard the tensor structure as a collection of isomorphisms of vector spaces  $V \otimes W \rightarrow V \otimes W$ , which is functorial with respect to  $V$  and  $W$ . This collection isomorphisms defines an element  $J \in (\mathcal{O}(G) \otimes \mathcal{O}(G))^*$  (see the preceding remarks to Lemma 4.1.7). It can be checked (similarly to Theorem 2.1 in [EG1]) that  $J$  is a Hopf 2-cocycle, and that  $\phi$  induces an isomorphism of cotriangular Hopf algebras  $A \rightarrow \mathcal{O}(G)^J$ . The implication (i)  $\Rightarrow$  (ii) is proved.

(ii)  $\Rightarrow$  (i).

We may assume that  $A = \mathcal{O}(G)^J$  (since  $S$  does not depend on the cotriangular structure). Since the categorical dimension of any object  $V \in \mathcal{C}$  does not change under twisting, we have



$\dim_c(V) = \dim(V)$  in the category of finite-dimensional right comodules over  $A$ . Therefore, we have

$$\begin{aligned}
& \text{tr}(S^2|_{A_V}) \\
&= \sum_{X,Y \in \text{Irr}(\mathcal{C})} N_V(X,Y) \dim_c(X) \dim_c(Y) \\
&= \sum_{X,Y \in \text{Irr}(\mathcal{C})} N_V(X,Y) \dim(X) \dim(Y) \\
&= \dim(A_V).
\end{aligned}$$

Since any finite-dimensional subcoalgebra  $C$  of  $A$  has the form  $A_V$  (for  $V = C$ ), this completes the proof of the theorem. ■

### 4.3 Proof of Proposition 4.2.2

Let us first prove the proposition in the case when  $A, B$  are finite-dimensional.

We need the following standard theorem from noncommutative algebra, which can be found for example in [DK, Chapter 3].

Let  $\mathcal{A}$  be a finite-dimensional algebra. Let  $\text{Irr}(\mathcal{A})$  be the set of isomorphism classes of irreducible left  $\mathcal{A}$ -modules. For  $M \in \text{Irr}(\mathcal{A})$ , let  $P(M)$  be the projective cover of  $M$ .

**Theorem 4.3.1** (i) *Any finite-dimensional projective  $\mathcal{A}$ -module is a direct sum of  $P(M)$ 's.*  
(ii) *For any  $M \in \text{Irr}(\mathcal{A})$ , the multiplicity of  $P(M)$  in the regular representation  $\mathcal{A}$  is equal to  $\dim(M)$ .*

Now we prove the proposition (in the finite-dimensional case). Let  $F_A, F_B$  be the forgetful functors from the categories of right  $A$ -comodules and right  $B$ -comodules, respectively, to the category of vector spaces. All we need to show is that  $F_B \circ F$  is isomorphic to  $F_A$ . Indeed, we have  $\text{End}(F_A) = A^*$  and  $\text{End}(F_B \circ F) = B^*$ , so any isomorphism between these two functors will induce an isomorphism of coalgebras  $A \rightarrow B$ , which (as one can easily see) induces  $F$ .

The functor  $F_A$  is represented by the regular representation  $A^*$ , and  $F_B \circ F$  by  $F^{-1}(B^*)$ . So it suffices to prove that  $F^{-1}(B^*)$  is isomorphic to  $A^*$  as an  $A$ -comodule.

Since  $B^*$  is free, it is projective, so  $F^{-1}(B^*)$  is also projective (as projectivity, unlike freeness, is a categorical property). Thus, by Theorem 4.3.1, we have

$$A^* = \bigoplus_{M \in \text{Irr}(A^*)} \dim(M) P(M) \text{ and } F^{-1}(B^*) = \bigoplus_{M \in \text{Irr}(A^*)} x(M) P(M),$$

where  $x(M)$  are nonnegative integers. But since  $F$  preserves dimensions, we have for any  $M \in \text{Irr}(A^*)$ :

$$\dim(M) = \dim(F(M)) = \dim(\text{Hom}_{B^*}(B^*, F(M))) = \dim(\text{Hom}_{A^*}(F^{-1}(B^*), M)) = x(M).$$

This completes the proof of the proposition in the finite-dimensional case.

Now let us consider the infinite-dimensional case. For simplicity consider the case when  $A, B$  are countably dimensional (the general case is similar). Then  $A = \cup_{n \geq 1} A_n$ , where  $A_n$  are finite-dimensional coalgebras. Let  $F_n : \text{Comod}_{f.d.}(A_n) \rightarrow \text{Comod}_{f.d.}(B)$  be the restriction of  $F$ , and let  $B_n := \text{End}(\text{Forget} \circ F_n)^*$ , where  $\text{Forget}$  is the forgetful functor on  $B$ -comodules. It is clear that  $B = \cup_{n \geq 1} B_n$ .

It is clear from the above finite-dimensional proof that for any  $n$ , there exists an isomorphism of coalgebras  $\phi_n : A_n \rightarrow B_n$ , such that  $\phi_{n+1}|_{A_n} = \phi_n \circ \text{Ad}(a_n)^*$ , where  $a_n \in A_n^*$  is an invertible element (this follows from the fact that  $\phi_n$  comes from an isomorphism of functors). Since the map of the multiplicative groups  $Gr(A^*) \rightarrow Gr(A_n^*)$  is surjective (because so is the corresponding Lie algebra map), we can lift  $a_n$  to an invertible element of  $A^*$ . Abusing notation, we will denote this element also by  $a_n$ .

Define  $\psi_n := \phi_n \circ \text{Ad}(a_1^{-1} \cdots a_{n-1}^{-1})^*$ . Then  $\psi_{n+1}|_{A_n} = \psi_n$ , so  $\{\psi_n\}$  defines an isomorphism of coalgebras  $\psi : A \rightarrow B$  which induces a functor isomorphic to  $F$ . ■

## 5 Examples of Twisted Function Algebras

In this section we give examples of Hopf 2-cocycles for  $\mathcal{O}(G)$  for certain algebraic groups  $G$ . We will construct these cocycles in the form of linear endomorphisms of tensor products of any two finite-dimensional  $G$ -modules  $V, W$ , functorial in terms of  $V, W$  (cf. the discussion before Lemma 4.1.7). These examples can be generalized, as usual, using the fact that if  $J$  is a Hopf 2-cocycle for  $G$  and  $\phi : G \rightarrow G'$  is a homomorphism then  $(\phi \otimes \phi)(J)$  is a Hopf 2-cocycle for  $G'$ .

**Example 5.1** Let  $G$  be the group of translations of an affine space. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Clearly,  $\mathfrak{g}$  is abelian. Therefore, it is straightforward to verify that for any  $r \in \Lambda^2 \mathfrak{g}$ , the element  $J(h) := e^{hr/2}$  is a Hopf 2-cocycle for any  $h \in k$  (the exponential series terminates in any  $V \otimes W$  as the components of  $r$  are nilpotent, so we get a polynomial of  $h$ ). In this case  $u = 1$  and  $S^2$  is the identity. □

**Example 5.2** Let  $G$  be the group of affine transformations of the line. Its Lie algebra  $\mathfrak{g}$  is spanned by two elements  $X, Y$  such that  $[X, Y] = Y$ . Define

$$J(h) := \sum_{n \geq 0} \frac{h^n}{n!} X(X-1) \cdots (X-n+1) \otimes Y^n.$$

It is not difficult to check that this is a Hopf 2-cocycle. In this case  $u = 1 + hY + O(h^2)$ , so  $S^2$  is not the identity. Thus  $\mathcal{O}(G)^J$  is an example of a pseudoinvolutive but not involutive cointegral Hopf algebra. Such a Hopf algebra can even be cosemisimple: it is enough to naturally embed  $G$  into  $GL(2)$  and consider the cosemisimple Hopf algebra  $\mathcal{O}(GL_2)^J$ . □

**Example 5.3** For every nonnegative integer  $n$ , let  $S_n$  be the symmetric group of permutations of  $n$  symbols. For any  $s \in S_n$  and  $0 \leq m \leq n$ , and any Lie algebra  $\mathfrak{g}$ , let  $L_{s,m} : \mathfrak{g}^{\otimes n} \rightarrow U(\mathfrak{g})^{\otimes 2}$  be the linear map determined by

$$L_{s,m}(a_1 \otimes \cdots \otimes a_n) = a_{s(1)} \cdots a_{s(m)} \otimes a_{s(m+1)} \cdots a_{s(n)}.$$

Let  $X$  be the disjoint union of the sets  $X_n := S_{2n} \times \{0, \dots, 2n\}$  for  $n \geq 2$ . Let  $k[X]$  be the set of  $k$ -valued functions on  $X$ . For any  $f \in k[X]$ , and any Lie algebra  $\mathfrak{g}$ , define the function  $J_f : \Lambda^2 \mathfrak{g} \rightarrow U(\mathfrak{g})^{\otimes 2}[[h]]$ , by the formula

$$J_f(r, h) = 1 + hr/2 + \sum_{n \geq 2} h^n \sum_{(s,m) \in X_n} f(s, m) L_{s,m}(r^{\otimes n}). \quad (10)$$

It is easy to show that if  $J_f(r, h)$  is a twist for  $U(\mathfrak{g})[[h]]$ , then  $r$  must satisfy the classical Yang-Baxter equation (CYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \quad (11)$$

Conversely, let  $r$  be a solution of CYBE.

**Definition 5.4** We say that  $f \in k[X]$  is a quantization function for  $r$  if the element  $J_f(r, h)$  is a twist. We say that  $f$  is a universal quantization function if this is the case for all  $\mathfrak{g}, r$ .

It is easy to construct concrete examples of quantization functions. For instance, in Example 5.1, it is straightforward to verify that  $e^{hr/2}$  comes from a quantization function for any  $r$  and abelian  $\mathfrak{g}$ . The existence of universal quantization functions is not obvious, but it follows from the quantization theory of [EK1]. Namely, a construction of such functions can be obtained from formula (3.1) in [EK1].

Quantization functions allow one to construct a large family of examples of Hopf 2-cocycles. Namely, we have:

**Theorem 5.5** Suppose that  $G$  is an algebraic group with Lie algebra  $\mathfrak{g}$ , and let  $N$  be its unipotent radical with Lie algebra  $\mathfrak{n}$ . Suppose that  $r$  is an element of  $\mathfrak{g} \wedge \mathfrak{n} \subset \Lambda^2 \mathfrak{g}$  which satisfies the CYBE. Then for any  $f \in k[X]$ , and any two algebraic representations  $V, W$  of  $G$ ,  $J_f(r, h)|_{V \otimes W}$  is a polynomial in  $h$  (i.e. the series terminates). In particular, for any  $h \in k$ ,  $J_f(r, h)$  is a well defined element in  $(\mathcal{O}(G) \otimes \mathcal{O}(G))^*$ . This element is a Hopf 2-cocycle for  $\mathcal{O}(G)$  if  $f$  is a quantization function for  $r$ .

**Proof:** We only need to show that the series  $J_f(r, h)$  terminates. Let  $V, W$  be two finite-dimensional algebraic representations of  $G$ . Let  $B_V, B_W$  be the images of  $U(\mathfrak{g})$  in  $\text{End}(V)$  and  $\text{End}(W)$ , and let  $I_V, I_W$  be the nilpotent radicals of  $B_V, B_W$ . It is clear that under the action of  $G$  in  $V, W$ ,  $\mathfrak{n}$  maps into  $I_V, I_W$ . Let  $n$  be a positive integer such that  $I_V^n = I_W^n = 0$ . Then it is clear from the definition that the series  $J_f(r, h)|_{V \otimes W}$  is a polynomial in  $h$  of degree  $\leq 2n - 2$ . ■

## 6 The Unipotency Conjecture

We conclude the paper with the following

**Conjecture 6.1** *For any pro-algebraic group  $G$  and Hopf 2-cocycle  $J$  for  $\mathcal{O}(G)$  over  $k$ , the operator  $S^2$  is unipotent on  $\mathcal{O}(G)^J$ .*

**Remark 6.2** We know that the sum of the eigenvalues of  $S^2$  on any finite-dimensional subcoalgebra is equal to its dimension, but the conjecture says that furthermore all of these eigenvalues are 1.  $\square$

**Remark 6.3** Conjecture 6.1 is obviously satisfied in Examples 5.1 and 5.2. Moreover, it follows from the theorem below that it is satisfied in Example 5.3 as well.  $\square$

Let  $\Sigma$  be an irreducible affine algebraic curve with a marked smooth point 0, and let  $\mathcal{O}(\Sigma)$  be the ring of regular functions on  $\Sigma$ . The standard example is  $\Sigma := k$ ,  $\mathcal{O}(\Sigma) = k[x]$ .

Let  $J : \mathcal{O}(G)^{\otimes 2} \rightarrow \mathcal{O}(\Sigma)$  be a family of Hopf 2-cocycles for  $\mathcal{O}(G)$  parametrized by  $a \in \Sigma$ , with  $J(0) = 1$ . In this case, we will say that  $J(a)$ , for any  $a \in \Sigma$ , is obtained by deformation of  $J(0)$ .

**Theorem 6.4** *Let  $J$  be obtained by deformation of 1. Then Conjecture 6.1 holds for the cotriangular Hopf algebra  $\mathcal{O}(G)^J$ .*

The rest of the section is devoted to the proof of the theorem.

To prove the theorem, we will choose a local parameter  $h$  on  $\Sigma$ , and write  $J$  as a formal power series in  $h$ :  $J = 1 + \sum_{n \geq 1} h^n r_n$ ,  $r_n \in (\mathcal{O}(G)^{\otimes 2})^*$ . We will say that  $J$  is *local* if  $r_n \in U(\mathfrak{g})^{\otimes 2}$  for all  $n$ .

**Lemma 6.5** *The Hopf 2-cocycle  $J$  is gauge equivalent to a local Hopf 2-cocycle. That is, there exists a “gauge transformation”  $g := 1 + hg_1 + h^2g_2 + \dots$ ,  $g_i \in \mathcal{O}(G)^*$ ,  $\varepsilon(g_i) = 0$ , such that the Hopf 2-cocycle  $J^g := \Delta(g)J(g^{-1} \otimes g^{-1})$  is local.*

**Proof:** Let us prove the statement modulo  $h^{n+1}$  by induction in  $n$ . The base of induction ( $n = 0$ ) is clear. To do the inductive step, assume that  $J$  is local modulo  $h^n$ . Observe that since  $r_n$  satisfies the Hopf 2-cocycle condition it follows that

$$r_n^{12} + (\Delta \otimes I)(r_n) - r_n^{23} - (I \otimes \Delta)(r_n) = f(r_1, \dots, r_{n-1}),$$

where  $f$  is a polynomial. Thus, we have  $dr_n \in U(\mathfrak{g})^{\otimes 3}$ , where  $d$  is the differential in the Hochschild complex of the coalgebra  $\mathcal{O}(G)^*$  with trivial coefficients.

It is well known that the embedding of coalgebras  $U(\mathfrak{g}) \rightarrow \mathcal{O}(G)^*$  defines an isomorphism of Hochschild cohomology of these coalgebras with trivial coefficients, and that both cohomology spaces are equal to  $\Lambda\mathfrak{g}$ , with the usual grading (the fact that the cohomology of  $U(\mathfrak{g})$  is  $\Lambda\mathfrak{g}$  is discussed for example in [Dr1, p.1435]). Therefore, there exists  $r'_n \in U(\mathfrak{g})^{\otimes 2}$  such that  $dr'_n = dr_n$ . Let  $s := r_n - r'_n$ . Then  $ds = 0$ . Therefore, we have  $s = s_0 + dz = s_0 + \Delta(z) - z \otimes 1 - 1 \otimes z$ , where  $s_0 \in \Lambda^2\mathfrak{g}$  and  $z \in \mathcal{O}(G)^*$ . Let us replace  $J$  with  $J^g$  for  $g := 1 + h^n(z - \epsilon(z))$ . Then  $J^g$  is local modulo  $h^{n+1}$ . The lemma is proved. ■

Now let us continue the proof of the theorem. By Lemma 6.5, we can assume that  $J(h)$  is local. Then  $J(h)$  is a twist for  $U(\mathfrak{g})[[h]]$ , so one can define the triangular quantized universal enveloping (QUE) algebra  $U(\mathfrak{g})[[h]]^{J(h)}$ . It is sufficient to show that the Drinfeld element  $u$  of this QUE algebra is unipotent on every finite-dimensional representation of  $\mathfrak{g}$ .

Using the main result of [EK2] (in the triangular case), we conclude that  $U(\mathfrak{g})[[h]]^{J(h)}$  is isomorphic to  $U_h(\mathfrak{g}, r)$ , where  $U_h$  is the quantization functor from [EK2], and  $r \in \Lambda^2\mathfrak{g}[[h]]$  is a solution of CYBE.

The QUE algebra  $U_h(\mathfrak{g}, r)$ , by definition, is obtained by twisting  $U(\mathfrak{g})$  using a twist  $J_f(r, h)$ , where  $f$  is a universal quantization function. This implies that the element  $u$  in this QUE algebra has the form

$$u = 1 + \sum_{n \geq 1} \sum_{\sigma \in S_{2n}} c_{n,\sigma} m_{2n}(\sigma \circ r^{\otimes n}),$$

where  $m_{2n}$  is the multiplication of  $2n$  components.

Now we show the unipotency of  $u$  in any finite-dimensional  $\mathfrak{g}$ -module by induction in the rank of  $r$ . Without loss of generality, we can assume that  $\mathfrak{g}$  is spanned by the components of  $r$ , and that  $V$  is an irreducible faithful  $\mathfrak{g}$ -module (if it is not faithful, we can go to  $r$ -matrices of smaller rank, for which the statement is known). This implies that  $\mathfrak{g}$  is reductive.

Now we will use the following theorem of Drinfeld (see e.g. [ES, Proposition 5.2]):

**Theorem 6.6** *Let  $\mathfrak{g}$  be a Lie algebra and  $r \in \Lambda^2\mathfrak{g}$  be a solution of CYBE whose components span  $\mathfrak{g}$ . If  $\mathfrak{g}$  is reductive, then it is abelian.*

This theorem immediately implies Theorem 6.4: if  $\mathfrak{g}$  is abelian then, because of the skew-symmetry of  $r$ , we have  $u = 1$ .

The rest of this section is the proof of Theorem 6.6, which we give for the reader's convenience.

It is clear that  $r$  defines a nondegenerate skew-symmetric bilinear form on  $\mathfrak{g}^*$ , hence one can define a skew-symmetric form  $r^{-1}$  on  $\mathfrak{g}$ . Since  $r$  satisfies CYBE, this form is a 2-cocycle.

Let  $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a$  be the splitting of  $\mathfrak{g}$  into the semisimple and the abelian parts. Assume that  $\mathfrak{g}_s \neq 0$ . Then  $H^2(\mathfrak{g}, k) = H^2(\mathfrak{g}_a, k) = \Lambda^2\mathfrak{g}_a^*$ , which shows that  $r^{-1}$  can be written as  $r^{-1}(x, y) = f([x, y]) + \rho(x, y)$ , where  $\rho \in \Lambda^2\mathfrak{g}_a^*$ ,  $f \in \mathfrak{g}^*$ . Thus, the decomposition  $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a$  is orthogonal under  $r^{-1}$ , and hence the form  $f([x, y])$  has to be nondegenerate on  $\mathfrak{g}_s$ .

But  $\mathfrak{g}_s$  has a nondegenerate invariant form  $(\cdot, \cdot)$ , and we can write the functional  $f(x)$  as  $(z, x)$  for some  $z \in \mathfrak{g}_s$ . Thus, the form  $(z, [x, y])$  should be nondegenerate. However, this is impossible as  $z$  obviously belongs to the kernel of this form (since  $(z, [z, y]) = 0$ ). Thus, we have a contradiction and hence  $\mathfrak{g}_s = 0$ . ■

## 7 Questions

In conclusion let us discuss possible directions for future research. The main remaining problem is to obtain a classification of twisted group algebras. We believe that this should be done by generalizing the techniques of [M] and [EG2] to the infinite-dimensional case and combining them with the techniques of [EK1, EK2]. Let us formulate some precise questions which are related to this problem.

**Question 7.1** Is any Hopf 2-cocycle for  $\mathcal{O}(G)$  gauge equivalent to a deformation of a Hopf 2-cocycle of finite rank?

**Question 7.2** Say that a cotriangular Hopf algebra is *minimal* if the R-form is nondegenerate. Now suppose that  $A$  is a minimal pseudoinvolutive cotriangular Hopf algebra, and that the underlying group  $G$  is reductive. Is it true that the connected component of the identity in  $G$  is abelian (i.e. a torus)?

**Remark 7.3** The answer to the classical analog of Question 7.2 is positive: if a Lie algebra spanned by the components of a skew-symmetric solution of CYBE is reductive, then it is abelian (see Theorem 6.6). □

**Question 7.4** Let  $A$  be any cotriangular Hopf algebra over  $k$ . Is it true that the eigenvalues of the square of its antipode are all roots of unity? Are all  $\pm 1$ ? Is it true at least for twisted function algebras?

Note that a positive answer to either form of Question 7.4 will imply Conjecture 6.1.

## References

- [BFM] Y. Bahturin, D. Fischman and S. Montgomery, Bicharacters, Twistings, and Schunert's Theorem For Hopf Algebras, *preprint*, 2000.
- [CWZ] M. Cohen, S. Westreich and S. Zhu, Determinants, integrality and Noether's theorem for quantum commutative algebras, *Israel J. of Math.* **96** (1996), 185-222.

- [De] P. Deligne, Categories Tannakiennes, In The Grothendick Festschrift, Vol. II, *Prog. Math.* **87** (1990), 111-195.
- [DM] P. Deligne and J. Milne, Tannakian Categories, *Lecture Notes in Mathematics* **900**, 101-228, 1982.
- [Do] Y. Doi, Braided bialgebras and quadratic bialgebras, *Comm. Algebra* **21** (1993), 1731-1749.
- [Dr] V. Drinfeld, On Almost Cocommutative Hopf Algebras, *Leningrad Mathematics Journal* **1** (1990), 321-342.
- [Dr1] V. Drinfeld, Quasi-Hopf algebras, *Leningrad Mathematics Journal* **1** (1990), 1419-1457.
- [DK] Y. A. Drozd and V. V. Kirichenko, Finite-dimensional algebras, *Springer*, New York, 1994.
- [EG1] P. Etingof and S. Gelaki, Some Properties of Finite-Dimensional Semisimple Hopf Algebras, *Mathematical Research Letters* **5** (1998), 191-197.
- [EG2] P. Etingof and S. Gelaki, The Classification of Triangular Semisimple and Cosemisimple Hopf Algebras Over an Algebraically Closed Field, *International Mathematics Research Notices*, to appear, math.QA/9905168.
- [EK1] P. Etingof and D. Kazhdan, Quantization of Lie Bialgebras, I, *Selecta Mathematica* **2** (1996), Vol.1, 1-41.
- [EK2] P. Etingof and D. Kazhdan, Quantization of Lie Bialgebras, II, *Selecta Mathematica* **4** (1998), 213-231.
- [ES] P. Etingof and O. Schiffmann, Lectures on Quantum Groups, Lectures in Mathematical Physics, *International Press, Boston, MA* (1998).
- [K] C. Kassel, Quantum Groups, *Springer*, New York, 1995.
- [LR] R. G. Larson and D. E. Radford, Semisimple Cosemisimple Hopf Algebras, *American J. of Mathematics* **110** (1988), 187-195.
- [M] M. Movshev, Twisting in group algebras of finite groups, *Func. Anal. Appl.* **27** (1994), 240-244.
- [R] D. E. Radford, The order of the antipode of a finite-dimensional Hopf algebra is finite, *Amer. J. Math.* **98** (1976), 333-355.